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On the reduction of n-fold tensor products in SU(2)

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Abstract. A map carrying irreducible representations of SU(2) into n-fold tensor product spaces of SU(2) is constructed. It is shown that the multiplicity of an SU(2) representation in the n-fold tensor product space is given by certain Gelfand patterns.

1. Introduction

It is well known that when the tensor product of two irreducible representations of SU(2) is decomposed into a direct sum of irreducible representations, the resulting decomposition is multiplicity free. The coefficients that transform between a tensor product basis and a direct sum basis, the so-called Wigner coefficients, can be written in closed form (see e.g. Hamermesh 1962; for a more technical discussion see e.g. Barut and Raczka 1977). However, when the tensor product of more than two representations is decomposed into a direct sum, multiplicity appears and it is necessary to find some means by which to distinguish between equivalent representations. The usual way is to couple representations in a stepwise manner until all the representations in the tensor product have been coupled together. Then the labels necessary to resolve the multiplicity are the intermediate irreducible representation labels, and the Wigner coefficients are sums of products of two-fold Wigner coefficients.

However, a problem with such an approach is that it does not preserve the symmetry that often appears in an n-fold tensor product space. For example, if one wishes to compute the Wigner coefficients for the n-fold tensor product $j\otimes\ldots\otimes j$, it is clear that the overall Wigner coefficients should preserve an S_n symmetry obtained by permuting the various factors in the tensor product. Such symmetry appears, for example, when computing the states of SU(3) in an SO(3) basis, in which case the relevant Wigner coefficients are those appearing in the n-fold tensor product $1\otimes\ldots\otimes 1$ (Klink 1983).

The primary goal of this paper will be to construct a map from an irreducible representation space of SU(2) to an arbitrary n-fold tensor product space in such a way that the multiplicity is labelled by irreducible representations of an underlying symmetric group. All the representations are treated on an equal footing in the n-fold tensor product space, without introducing intermediate angular momenta as multiplicity labels. The representations of SU(2) will be given in terms of polynomials over GL(2, $\mathbb{C}$). Since one does not normally realise the representations of SU(2) in this

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way, § 2 will review some of the more important properties of such polynomial representations, as well as discussing the meaning of n-fold tensor product spaces as polynomials over the n-fold direct product of GL(2, ℂ). The actual form and properties of the map such as orthogonality properties are discussed in § 3; the appendix shows how certain matrix elements of the symmetric group needed in the computation of Wigner coefficients can be obtained. Section 4 deals with an example, namely how to construct the map carrying irreducible representations into the tensor product space $1\otimes1\otimes1/2\otimes3/2$.

One of the main results of this paper is that the multiplicity of an SU(2) representation in the n-fold tensor product space is given by certain Gelfand patterns, arising from an underlying symmetric group. What we wish to show in the case of SU(2) is that the interplay between representations of this underlying symmetric group and polynomial representations of SU(2) provides a natural setting in which to compute many of the coefficients needed in the application of group theory to physics.

2. n-fold tensor products from a holomorphic induction point of view

All the irreducible representations of SU(2) will be realised as polynomials over GL(2, ℂ) (Klink and Ton-That 1979); that is, an irreducible representation space for the representation $(m)\equiv(m_1, m_2)$ of GL(2, ℂ) is given by

$$V^{(m)}=\{f:\mathrm{GL}(2,\mathbb{C})\to\mathbb{C}, f(bg)=b_{11}^{m_1}b_{22}^{m_2}f(g)\},\qquad f \text{ polynomial in } \mathrm{GL}(2,\mathbb{C}). \tag{1}$$

Here b is an element of the Borel subgroup B of lower triangular matrices, $\mathrm{B}=\{\left(\begin{smallmatrix}b_{11} & 0\\ b_{21} & b_{22}\end{smallmatrix}\right)\}$, and $m_1\geqslant m_2$ are integers. An irreducible representation is then given by

$$(T_{g_0}f)(g)=f(gg_0),\qquad f\in V^{(m)}, g_0\in\mathrm{GL}(2,\mathbb{C}). \tag{2}$$

If g_0 is restricted to the subgroup SU(2), the representation space $V^{(m)}$ remains irreducible. However, representations related by (m_1+k, m_2+k), k an integer, are now equivalent, so that if k is chosen as $-m_2$, all inequivalent representations of SU(2) can be written as $(m, 0)$. These representations are related to the usual angular momentum by $m=2j$.

A natural 'differentiation' inner product exists on $V^{(m)}$, given by

$$(f,f')\equiv f(\partial/\partial g_{ij})\overline{f'(\overline{g_{ij}})}|_{g=0},\qquad f,f'\in V^{(m)}, \tag{3}$$

and if g_0 is restricted to SU(2), T_{g_0} is unitary; i.e. $(T_{g_0}f, T_{g_0}f')=(f,f')$.

An orthogonal basis for $V^{(m)}$ is given by

$$h_k^{(m)}(g)=g_{11}^{k-m_2}g_{12}^{m_1-k}|g|^{m_2} \tag{4}$$

where $m_1\geqslant k\geqslant m_2$ (the Gelfand–Cetlin betweenness relations) and $|g|$ is the determinant of $g\in\mathrm{GL}(2,\mathbb{C})$. The basis polynomials (4) are not normalised; in fact, non-normalised polynomials will be used in some intermediate calculations because the maps needed to obtain Wigner coefficients do not preserve the norm of the polynomials with respect to the inner product (3).

We will consistently denote the unnormalised bases by $h_k^{(m)}$ and normalised polynomial basis elements by $e_k^{(m)}$. If $m_2=0$ in (4), the following polynomial realisation for $|j, j_3\rangle$ results:

$$|j, j_3\rangle \to e_k^{(m,0)}(g) = \frac{g_{11}^k g_{12}^{m-k}}{[k!(m-k)!]^{1/2}}, \qquad j = \frac{m}{2}, j_3 = k - \frac{m}{2}. \tag{5}$$

From these definitions it follows that an n-fold tensor product space $T^n \equiv V^{(m_1)} \otimes \ldots \otimes V^{(m_n)}$, (m_i), $i = 1 \ldots n$, arbitrary representations of SU(2), is given by

$$T^n = \{F\colon \mathrm{GL}(2, \mathbb{C}) \times \ldots \times \mathrm{GL}(2, \mathbb{C}) \to \mathbb{C}, \qquad F \text{ polynomial}, \\ F(b_1 g_1, \ldots, b_n g_n) = \pi^{(m_1)}(b_1) \ldots \pi^{(m_n)}(b_n) F(g_1, \ldots, g_n)\}, \tag{6}$$

where $\pi^{(m)}(b) = b_{11}^m$, $b_i \in B$, $g_i \in \mathrm{GL}(2, \mathbb{C})$, $i = 1 \ldots n$. Here $(m_i) = (m_i, 0)$ because we are considering only SU(2) representations.

The goal of this paper is to find the map that carries a representation space $V^{(M)}$ into T^n. To that end we define an auxiliary space of r-fold tensor products of the fundamental representations (10):

$$T^r_{(10)} \equiv V^{(10)} \otimes \ldots \otimes V^{(10)}. \tag{7}$$

The map carrying $V^{(M)}$ to T^n will be composed from two maps, namely $\alpha_\eta^{(M)}\colon V^{(M)} \to T^r_{(10)}$ and $\Phi\colon T^r_{(10)} \to T^n$. $\alpha_\eta^{(M)}$ will be discussed in the next section. Φ is defined by

$$(\Phi F)(g_1, \ldots, g_n) = F(\overbrace{g_1, \ldots, g_1}^{m_1}, \overbrace{g_2, \ldots, g_2}^{m_2}, \ldots, \overbrace{g_n, \ldots, g_n}^{m_n}), \qquad F \in T^r_{(10)}. \tag{8}$$

The numbers above the GL(2, $\mathbb{C}$) arguments, $m_1 \ldots m_n$, come from the representations $(m_i 0)$, $i = 1 \ldots n$, of the original tensor product space T^n, and $r = \Sigma_{i=1}^n m_i$. The map Φ takes functions $F(g_1, \ldots, g_r)$ of $T^r_{(10)}$ and sets the first m_1 arguments equal, the next m_2 arguments equal and so forth, so that finally ΦF is a polynomial function of only $g_1 \ldots g_n$ arguments of GL(2, $\mathbb{C}$).

To show that ΦF is in T^n, it is necessary to check that the conditions of (6) are satisfied. But from the definition of Φ, only the covariance condition is not obvious; we must check

$$\begin{aligned}
(\Phi F)&(b_1 g_1, \ldots, b_n g_n) \\
&= F(\overbrace{b_1 g_1, \ldots, b_1 g_1}^{m_1}, \overbrace{b_2 g_2, \ldots, b_2 g_2}^{m_2}, \ldots, \overbrace{b_n g_n, \ldots, b_n g_n}^{m_n}) \\
&= \overbrace{\pi^{(10)}(b_1) \ldots \pi^{(10)}(b_1)}^{m_1} \overbrace{\pi^{(10)}(b_2) \ldots \pi^{(10)}(b_2)}^{m_2} \ldots \overbrace{\pi^{(10)}(b_n) \ldots \pi^{(10)}(b_n)}^{m_n} \\
&\quad \times (\Phi F)(g_1, \ldots, g_n) \\
&= \pi^{(m_1 0)}(b_1) \pi^{(m_2 0)}(b_2) \ldots \pi^{(m_n 0)}(b_n) (\Phi F)(g_1, \ldots, g_n)
\end{aligned}$$

so that indeed $(\Phi F)(g_1, \ldots, g_n)$ is in T^n.

Since Φ is defined by setting many of the arguments of $F \in T^r_{(10)}$ equal to one another, it follows that for a basis element like $e_{k_1}^{(10)} \otimes \ldots \otimes e_{k_r}^{(10)} \in T^r_{(10)}$, nothing is changed in $\Phi(e_{k_1}^{(10)} \otimes \ldots \otimes e_{k_r}^{(10)})(g_1, \ldots, g_n)$ under the interchange of the first m_1 labels $k_1 \ldots k_{m_1}$, or the next m_2 labels, etc. This fact will be of importance when determining the meaning of the multiplicity label η of the map $\alpha_\eta^{(M)}$ of $V^{(M)}$ to $T^r_{(10)}$, to which we now turn.

3. The map from $V^{(M)}$ to $T^r_{(10)}$

The index η on the map $\alpha^{(M)}_\eta$ from $V^{(M)}$ to $T^r_{(10)}$ is needed to label the different ways in which the irreducible representation (M) of SU(2) sits in $T^r_{(10)}$. To define η we note that $T^r_{(10)}$ actually carries representations of $\mathrm{SU}(2)\times \mathrm{S}_r$, where S_r is the permutation group on r numbers. The representation of S_r in $T^r_{(10)}$ is given by

$$T_p(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r}) = e^{(10)}_{p(k_1)}\otimes\ldots\otimes e^{(10)}_{p(k_r)}, \tag{9}$$

where $p(k_i)$ is the permutation of the ith index by $p\in \mathrm{S}_r$. Then the representation of $\mathrm{SU}(2)\times\mathrm{S}_r$ on basis elements in $T^r_{(10)}$ is given by

$$\begin{aligned} T_{(g_0,p)}(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r})(g_1,\ldots,g_r) &= (e^{(10)}_{p(k_1)}\otimes\ldots\otimes e^{(10)}_{p(k_r)})(g_1g_0,\ldots,g_rg_0)\\ &= (e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r})(p^{-1}(g_1g_0),\ldots,p^{-1}(g_rg_0)). \end{aligned} \tag{10}$$

Weyl (as discussed in Robinson (1961)) has shown that $T^r_{(10)}$ contains only those representations of $\mathrm{SU}(2)\times\mathrm{S}_r$ of the form $(M)\equiv(M_1,M_2)$, where $M_1+M_2=r$, $M_1\geqslant M_2$. That (M) is a representation of SU(2) was shown in § 2. On the other hand, all irreducible representations of S_r are given by Young diagrams (Hamermesh 1962, Robinson 1961), and the irreducible representations of interest here are just those with M_1 boxes in the first row and M_2 boxes in the second row. Thus, (M) labels an irreducible representation of both SU(2) and S_r and only those representations of $\mathrm{SU}(2)\times\mathrm{S}_r$ with $M_1+M_2=r$, $M_1\geqslant M_2$ can occur in $T^r_{(10)}$, all with multiplicity one.

Let $e^{(M)}_k$ be a basis element of SU(2), of the form $h^{(M)}_k/\|h^{(M)}_k\|$, where $h^{(M)}_k$ is defined in (4), and let $f^{(M)}_\eta$ be a basis element in an irreducible representation space $W^{(M)}$ of S_r. Define a map $\alpha: V^{(M)}_{\mathrm{SU}(2)}\otimes W^{(M)}_{\mathrm{S}_r}\to T^r_{(10)}$ by

$$\alpha(e^{(M)}_k\otimes f^{(M)}_\eta) = \sum_{k_1,\ldots,k_r}\langle(10)k_1\ldots(10)k_r|(M)k\eta\rangle\times e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r}, \tag{11}$$

where $\langle|\rangle$ are Wigner coefficients for the direct sum decomposition of $T^r_{(10)}$ into irreducible representations (M) of $\mathrm{SU}(2)\times\mathrm{S}_r$.

Since Wigner coefficients for multiplicity free decompositions are orthogonal, (11) can be inverted to give

$$e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r} = \sum_{(M)k\eta}\langle|\rangle\alpha(e^{(M)}_k\otimes f^{(M)}_\eta). \tag{12}$$

Equation (12) can be used to compute the Wigner coefficients. We first introduce a projection operator $P^{(M)}_\eta$ which projects out the basis element η of the representation (M) of S_r. (The actual form of $P^{(M)}_\eta$ is given in the last equation of the appendix.) Then

$$P^{(M)}_\eta(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r}) = \sum_k\langle|\rangle\alpha(e^{(M)}_k\otimes f^{(M)}_\eta). \tag{13}$$

Now define a map $\Phi_{(M)}$ from $T^r_{(10)}$ to polynomials over $\mathrm{GL}(2,\mathbb{C})$:

$$\begin{gathered}(\Phi_{(M)}F)(g) = F(\overbrace{g,g,\ldots,g}^{M_1},\overbrace{\pi g,\pi g,\ldots,\pi g}^{M_2}),\\ F\in T^r_{(10)},\qquad \pi=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in\mathrm{GL}(2,\mathbb{C}).\end{gathered} \tag{14}$$

Then

$$(\Phi_{(M)}F)(dg) = d_1^{M_1}d_2^{M_2}(\Phi_{(M)}F)(g), \tag{15}$$

where $d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$. $\Phi_{(M)}F$ does not transform to the left with respect to the Borel group B of (1), but only with respect to the diagonal subgroup of B. Therefore, $\Phi_{(M)}F$ is not in general an element of $V^{(M)}$. But $V^{(M)}$ is in $\Phi_{(M)}T^r_{(10)}$, because any element in $V^{(M)}$ satisfies (15).

So applying $\Phi_{(M)}$ as defined in (14) to both sides of (13) gives

$$\begin{aligned} &\Phi_{(M)}P^{(M)}_{\eta}(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r}) = \sum_{k} \langle|\rangle e^{(M)}_{k}, \\ &K^{(M)\,(10)\ldots(10)}_{k\eta\;\, k_1\ldots k_r} = (e^{(M)}_{k}, \Phi_{(M)}P^{(M)}_{\eta}(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r})), \end{aligned} \tag{16}$$

where $K^{\cdots}_{\cdots}$ is a Clebsch–Gordan coefficient (an unnormalised Wigner coefficient) given via the differentiation inner product (3). The reason the second line of (16) does not yield the Wigner coefficient $\langle|\rangle$ is that $\Phi_{(M)}$ does not in general preserve norms. That being the case, it is possible to lump all sorts of constants (which may depend on (M), k or η—such as $\|e^{(M)}_k\|$) in $K^{\cdots}_{\cdots}$, and then after having computed $K^{\cdots}_{\cdots}$, normalise to get $\langle|\rangle$. That is, after $K^{\cdots}_{\cdots}$ has been computed according to (16), for fixed (M), k, η, square and sum the entries in $k_1 \ldots k_r$. The square root of this normalisation factor divided into $K^{\cdots}_{\cdots}$ then gives the Wigner coefficients $\langle|\rangle$.

Assuming now that the Wigner coefficient $\langle|\rangle$ is known, we can return to the goal of this section, namely to define a map carrying $V^{(M)}$ to $T^r_{(10)}$:

$$\alpha^{(M)}_{\eta} e^{(M)}_{k} \equiv \sum_{k_1\ldots k_r} \langle|\rangle e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r}, \tag{17}$$

where η is now to be understood as a multiplicity label arising from basis elements of the (M) representation of S_r.

Apply the Φ map of (8) to both sides of (17). Then

$$(\Phi\alpha^{(M)}_{\eta} e^{(M)}_{k})(g_1, \ldots, g_n) = \sum_{k_1\ldots k_r} \langle|\rangle \times \Phi(e^{(10)}_{k_1}\otimes\ldots\otimes e^{(10)}_{k_r})(g_1, \ldots, g_n) \tag{18}$$

carries the basis element $e^{(M)}_k$ of $V^{(M)}$ into the tensor product space T^n. From (18) it is clear that the first m_1 polynomials $e^{(10)}_{k_1}(g_1) \ldots e^{(10)}_{k_{m_1}}(g_1)$ are symmetric in g_1, the next m_2 polynomials are symmetric in g_2 etc. But this means that only when the representations of the subgroup $S_{m_1} \times \ldots \times S_{m_n}$ are the symmetric (identity) representation will the map $\Phi\alpha^{(M)}_{\eta}$ be non-zero. If any of the representations of the groups S_{m_i}, $i = 1 \ldots n$, are not the identity representation, then the sum in (18) will give zero, because the Wigner coefficients will transform under the $k_1 \ldots k_r$ sum as a non-identity representation, while the polynomials $e^{(10)}_{k_1}(g_1) \ldots$ are symmetric under the interchange of these indices.

We conclude that the multiplicity η, the number of times (M) is contained in T^n, is given by the number of times the identity representation of the subgroup $S_{m_1} \times \ldots \times S_{m_n}$ is contained in the representation (M) of S_r. But as shown in Kramer *et al* (1981), this is nothing other than the number of Gelfand patterns that can be formed with $M_1M_20 \ldots 0$ at the top of the pattern and m_1 at the bottom. The betweenness relations demand that below $M_1M_20 \ldots 0$ a representation of S_{r-m_n} should fit, below that a representation of $S_{r-m_n-m_{n-1}}$ and so forth until the last entry of the pattern, namely m_1, results. An example of how such computations are made will be given in § 4.

We must now show that $\Phi\alpha^{(M)}_{\eta} e^{(M)}_{k}$ is orthogonal in T^n. In general $\Phi\alpha^{(M)}_{\eta} e^{(M)}_{k}$ will not be normalised since Φ does not preserve norms, but $\|\Phi\alpha^{(M)}_{\eta} e^{(M)}_{k}\|$ is readily computed using the differentiation inner product (3). Now for a fixed (M) we have seen that $\alpha(V^{(M)} \otimes W^{(M)})$ is irreducible in $\mathrm{SU}(2) \times S_r$. Since Φ intertwines with T_g, we conclude—using a theorem in Naimark (1964)—that $\Phi\alpha^{M()}_{\eta} e^{(M)}_{k}$ is orthogonal in T^n.

To conclude this section we compute the Wigner coefficients for (M) in T^n. These coefficients, not to be confused with $\langle|\rangle$, the Wigner coefficients of (M) in $T'_{(10)}$ given by (16), are obtained by writing a typical product of (10) basis elements as an (m_i) basis element. We illustrate with the case of (m_1):

$$P^{(m_1)}(e_{k_1}^{(10)}\otimes\ldots\otimes e_{k_{m_1}}^{(10)})(g)=\sum_k\langle(10)k_1,\ldots,(10)k_{m_1}|m_1k\rangle\alpha^{(m_1)}e_k^{(m_1)}, \tag{19}$$

where there is no η on the projection operator $P^{(m_1)}$ since the identity representation of S_{m_1} is one dimensional. Then from equation (18)

$$\begin{aligned}K_k^{(m_1)}{}_{k_1\ldots k_{m_1}}^{(10)\ldots(10)}&=(e_k^{(m_1)},\Phi_\pi P^{(m_1)}(e_{k_1}^{(10)}\otimes\ldots\otimes e_{k_{m_1}}^{(10)}))\\&=\frac{\partial^k}{\partial^k_{g_{11}}}\frac{\partial^{m_1-k}}{\partial^{m_1-k}_{g_{12}}}g_{11}^{k_1+\ldots+k_{m_1}}g_{12}^{m_1-(k_1+\ldots+k_{m_1})}|_{g=0}\\&=k!(m_1-k)!\delta_{k,k_1+\ldots+k_{m_1}}.\end{aligned} \tag{20}$$

For a fixed m_1 and k, these entries are all equal. Therefore, the Wigner coefficients

$$\langle(10)k_1,\ldots,(10)k_{m_1}|(m_1)k\rangle=(N^k_{k_1\ldots k_{m_1}})^{-1/2}\delta_{k,k_1+\ldots+k_{m_1}},$$

where $N^k_{k_1\ldots k_{m_1}}$ is the number of ways that k_i, $i=1\ldots m_1$, can be chosen either 0 or 1 so that $\Sigma_{i=1}^m k_i=k$. Then

$$\begin{aligned}\Phi\alpha_\eta^{(M)}e_k^{(M)}=&\sum_{k_1+\ldots+k_r=k}\langle(10)k_1,\ldots,(10)k_r|(M)k\eta\rangle\times\langle(10)k_1,\ldots,(10)k_{m_1}|m_1k_1\rangle\\&\times\ldots\times\langle(10)k_{r-m_n},\ldots,(10)k_{m_r}|m_nk_n\rangle\times e_{k_1}^{(m_1)}\otimes\ldots\otimes e_{k_n}^{(m_n)}\end{aligned}$$

which means that the Wigner coefficients are given (up to a normalisation factor) by

$$\begin{aligned}\langle(m_1)K_1,\ldots,(m_n)K_n|(M)k\eta\rangle=&\sum_{k_1+\ldots+k_r=k}\langle(10)k_1,\ldots,(10)k_r|(M)k\eta\rangle\\&\times\langle(10)k_1,\ldots,(10)k_{m_1}|(m_1)K_1\rangle\times\ldots\\&\times\langle(10)k_{r-m_n},\ldots,(10)k_{m_r}|m_nk_n\rangle,\end{aligned} \tag{21}$$

where the Wigner coefficients on the right-hand side of (21) are given by (16) and (20).

4. An example

To illustrate how the general formalism works, we discuss a simple example, namely the tensor product $1\otimes1\otimes1/2\otimes3/2$. The multiplicity is easily obtained by stepwise coupling the representations. The final result is

Representation	(M)	Multiplicity	Dimension
$j=4$	(8, 0)	1	$9\times1=9$
3	(7, 1)	3	$7\times3=21$
2	(6, 2)	5	$5\times5=25$
1	(5, 3)	5	$3\times5=15$
0	(4, 4)	2	$1\times2=2$
			72

(22)

The dimension of $T^4 = V^1 \otimes V^1 \otimes V^{1/2} \otimes V^{3/2} = V^{(20)} \otimes V^{(20)} \otimes V^{(10)} \otimes V^{(30)}$ is 72. From (7) we see that $r = 8$, so that $T^8_{(10)} = V^{(10)} \otimes \ldots \otimes V^{(10)}$ is an eight-fold tensor product of fundamental representations with an underlying S_8 symmetry. The direct sum decomposition of $T^8_{(10)}$ is given by the S_8 irreducible representations:

Representation	(M)	S_8 dimension	Dimension
$j = 4$	(8, 0)	1	$9\times 1 = 9$
3	(7, 1)	7	$7\times 7 = 49$
2	(6, 2)	20	$5\times 20 = 100$
1	(5, 3)	28	$3\times 28 = 84$
0	(4, 4)	14	$1\times 14 = 14$
			256

(23)

where 256 is the dimension of $T^8_{(10)}$.

The numbers in the 'S_8 dimension' column give the multiplicity of (M) in $T^8_{(10)}$, and are always greater than or equal to the corresponding multiplicity of T^4 in (22), indicating that the Φ map from $T^8_{(10)}$ to T^4 has reduced the multiplicity.

To obtain the multiplicity using the arguments following (18), we must find the number of times that the identity representation of $S_2 \times S_2 \times S_1 \times S_3$ is contained in the (M) representation of S_8. But this is given by a Gelfand pattern with (M) at the top of the pattern, a representation of S_5 with representations 'between' (M), next a representation of S_4 'between' the S_5 representation, and finally the symmetric representation 2 of S_2 'between' the representation of S_4. We get

Representation	(M)	Gelfand patterns					Multiplicity
$j = 4$	(8, 0)	8000					1
		500					
		40					
		2					
3	(7, 1)	7100	7100	7100			3
		500	410	410			
		40	40	31			
		2	2	2			
2	(6, 2)	6200	6200	6200	6200	6200	5
		500	410	410	320	320	
		40	40	31	31	22	
		2	2	2	2	2	
1	(5, 3)	5300	5300	5300	5300	5300	5
		500	410	410	320	320	
		40	40	31	31	22	
		2	2	2	2	2	
0	(4, 4)	4400	4400				2
		410	410				
		40	31				
		2	2				

(24)

and these multiplicity numbers agree with the multiplicity of (22) obtained by stepwise coupling.

To each of these Gelfand patterns η corresponds a matrix element $d^{(M)}_{\eta\eta}(p)$, $p \in S_8$, which is used in the projection operator $P^{(M)}_\eta$, needed to compute the Wigner coefficients which define the map $\alpha^{(M)}_\eta$. It is this connection between η as a multiplicity label and η as a label in the S_8 matrix element that makes it possible actually to compute the needed Wigner coefficients.

5. Conclusion

We have shown how to construct a map $\Phi\alpha^{(M)}_\eta$ carrying basis elements $e^{(M)}_k$ of the representation space $V^{(M)}$ of SU(2) into the n-fold tensor product space T^n. The multiplicity label η corresponds to the number of possible ways the representation (M) of S_r contains the identity representation of the subgroup $S_{m_1}\times\ldots\times S_{m_n}$ of S_r. But this in turn is given by Gelfand patterns, as discussed in the appendix. These Gelfand patterns also specify matrix elements of S_r needed in the actual computation of the $\alpha^{(M)}_\eta$ map.

The composition map $\Phi\alpha^{(M)}_\eta$ is given through Wigner coefficients calculated from (14) with the help of the S_r matrix elements. Thus $(\Phi\alpha^{(M)}_\eta e^{(M)}_k)(g_1, \ldots, g_n)$ is a polynomial in the space T^n involving n GL(2, $\mathbb{C}$) variables and the Wigner coefficients (22). Rather than try to find a closed form representation for such polynomials, it makes more sense to program a computer to compute the S_r matrix elements and then differentiate the polynomials needed to get the Wigner coefficients. In fact, this seems to be the general situation when using holomorphic induction techniques. Representations of the compact groups are given in terms of certain polynomial variables coming from the complexification of the compact group, and the coefficients such as Wigner or Racah coefficients involve projecting out certain polynomials and differentiating them.

For example, once the polynomials $\Phi\alpha^{(M)}_\eta e^{(M)}_k$ are known (equation (21)), it is possible to compute Racah coefficients; Racah coefficients are normally defined as the overlap coefficients between two different stepwise coupling schemes, and can be shown to be sums of products of twofold Wigner coefficients. But $\Phi\alpha^{(M)}_\eta e^{(M)}_k$ does not rely on a stepwise coupling procedure. Hence, if $e^{(M)}_\gamma$ is a polynomial basis in T^n obtained by stepwise coupling, the numbers $(e^{(M)}_\gamma, \Phi\alpha^{(M)}_\eta e^{(M)}_k)$ will give the overlap between the stepwise scheme γ and the scheme η; the coefficients (,) are again given by differentiating polynomials, using (3). Racah coefficients giving the overlap between two different stepwise schemes are then given with respect to the standard scheme $\Phi\alpha^{(M)}_\eta e^{(M)}_k$, as is the case with the Poincaré group (Klink 1975).

Finally it should be pointed out that in the n-fold tensor product $j_1\otimes\ldots\otimes j_n$, no further symmetries due to the interchange of identical representations appearing in the tensor product have been taken into account. But for tensor products such as $j\otimes\ldots\otimes j$, new symmetries arise precisely from such possible interchange. If one demands certain symmetry types, the multiplicity calculated in § 3 will, of course, change; here the notion of plethysm (Littlewood 1950, Wybourne 1970) appears, a topic which will be discussed in a succeeding paper. For example, if one requires in the tensor product $1\otimes1\otimes1/2\otimes3/2$ discussed in § 2 that only the symmetric part of $1\otimes1$ appear, then the multiplicity of (22) reduces to 1, 2, 3, 3, 1, respectively.

Although only SU(2) has been analysed in this paper, the ideas used all generalise to the SU(m) (and even to the SO(m) and Sp($2m$)) groups. In succeeding papers we will show how the interplay between the representations of the symmetric group, supplying the multiplicity labels, and the polynomial character of the representations of SU(m) allow one to decompose arbitrary n-fold tensor product of representations in arbitrary (i.e., in general, non-Gelfand) bases.

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Appendix

The space $T^r_{(10)}$ (equation (7)) has associated with it an underlying group S_r, whose irreducible representations (M) are related to the angular momentum j of the representations being mapped into T^n by $M_1+M_2=r$, $M_1-M_2=2j$, with $r=\Sigma_{i=1} m_i$. The dimension of the representation (M) of S_r gives the multiplicity of the representation (M) of SU(2) in $T^r_{(10)}$. A basis in the (M) representation of S_r is given by the labels of a subgroup of S_r; a natural choice in light of the Φ map is the subgroup $S_{m_1}\times\ldots\times S_{m_n}$ of S_r, for when the Φ map is composed with $\alpha^{(M)}_\eta$, only the identity representation of $S_{m_1}\times\ldots\times S_{m_n}$ survives. Hence, the multiplicity label η refers to the different ways in which the representation (M) of S_r contains the identity representation of the subgroup $S_{m_1}\times\ldots\times S_{m_n}$. Kramer *et al* (1981) show that η is equivalent to certain Gelfand patterns; here we simply summarise those facts that are needed.

The numbers $m_1,\ldots,m_n$ are used to define a Gelfand pattern in the following way: we have first that $m_1+\ldots+m_n=r$. Take away m_n, then $m_1+\ldots+m_{n-1}=r_1$ defines a subgroup S_{r_1}, $m_1+\ldots+m_{n-2}=r_2$ a subgroup S_{r_2}, until finally only m_1 is left. The chain of subgroups $S_r>S_{r_1}>\ldots>S_{m_1}$ defines a Gelfand pattern of possible irreducible representations of the subgroups, with (M), the irreducible representation of S_r, at the top and m_1, the identity representation of S_{m_1}, at the bottom of the pattern. Each representation of S_{r_i} is given by a sequence of decreasing integers, and these integers must satisfy the betweenness relations with respect to the preceding representation. In the example of § 4, the tensor product $1\otimes1\otimes1/2\otimes3/2$ gives $m_1=m_2=2$, $m_3=1$ and $m_4=3$, so that the identity representation of $S_2\times S_2\times S_1\times S_3$ in (M) of S_8 is needed. The chain of subgroups defining the Gelfand pattern is then $S_8>S_5>S_4>S_2$ with $r_1=5$, $r_2=4$. The irreducible representation (71) of S_8 has dimension 7. This multiplicity is reduced by allowing only symmetric representations of $S_2\times S_2\times S_1\times S_3$. According to Kramer *et al* (1981) all such possibilities are given by the Gelfand patterns formed by the irreducible representations of the chain, as given in (24).

Once the Gelfand pattern η is given it is also possible to compute the matrix element $d^{(M)}_{\eta\eta}(p)$, $p\in S_r$, needed for the projection operator $P^{(M)}_\eta$, equation (12). Since only the identity representation of $S_{m_1}\times\ldots\times S_{m_n}$ is allowed, the double cosets of S_r with respect to this subgroup will greatly simplify the calculation of the matrix elements $d^{(M)}_{\eta\eta}(p)$, for if $p\in S_r$, then we can write $p=hp_{ik}\tilde{h}$, where $p_{ik}\in S_r$ is a double coset representative, and $h,\tilde{h}\in S_{m_1}\times\ldots\times S_{m_n}$. Then $d^{(M)}_{\eta\eta}(p)=d^{(M)}_{\eta\eta}(p_{ik})$. As shown in Kramer *et al* (1981), the double cosets are in 1–1 correspondence with a set of non-negative integers k_{ij} satisfying $\Sigma_{j=1}^{n} k_{ij} = m_i$, $\Sigma_{i=1}^{n} k_{ij} = m_j$, and the double coset representatives can be obtained from these integers.

Kramer *et al* (1981) also show how to construct generating functions from matrix elements of GL(n, $\mathbb{C}$) to obtain the matrix elements $d_{\eta\eta}^{(M)}(p_{ij})$. In a previous publication (Klink and Ton-That 1982), we have shown how to compute the matrix elements of GL(n, $\mathbb{C}$) in a Gelfand–Cetlin basis using holomorphic induction techniques, so there is a natural way actually to obtain the S_r matrix elements. However, there are alternative ways of obtaining S_r matrix elements, and the method actually used, as discussed in the conclusion, will depend on how easily the required numbers can be obtained from a computer.

In whichever way the S_r matrix elements are obtained, the projection operator needed for computing the Wigner coefficients is then given (up to a factor which is lumped into the coefficient of $K^{\sqsubset}_{\sqsupset}$, equation (16)) by

$$P_{\eta}^{(M)} = \sum_{p \in S_r} d_{\eta\eta}^{(M)}(p_{ij}) T_p,$$

where T_p is defined in (9).

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